Lesson 8. Work Scheduling Models, Revisited

1 The postal workers problem, revisited

Example 1. Postal employees in Simplexville work for 5 consecutive days, followed by 2 days off, repeated weekly. Below are the minimum number of employees needed for each day of the week:

Day	Employees needed
Monday	7
Tuesday	8
Wednesday	7
Thursday	6
Friday	6
Saturday	4
Sunday	5

We want to determine the minimum total number of employees needed.

Our original model:

Decision variables. Let

$$
\begin{aligned}
x_1 &= \text{number of employees who work “shift 1” – i.e. Monday to Friday}\\
x_2 &= \text{number of employees who work “shift 2” – i.e. Tuesday to Saturday}\\
&\vdots\\
x_7 &= \text{number of employees who work “shift 7” – i.e. Sunday to Thursday}
\end{aligned}
$$

Objective function and constraints.

$$
\begin{array}{llr}
\min & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & \\
\text{s.t.} & x_1 + x_4 + x_5 + x_6 + x_7 \geq 7 & \text{(Mon)}\\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq 8 & \text{(Tue)}\\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq 7 & \text{(Wed)}\\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq 6 & \text{(Thu)}\\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq 6 & \text{(Fri)}\\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq 4 & \text{(Sat)}\\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 5 & \text{(Sun)}\\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7 \geq 0 &
\end{array}
$$

- Left hand side of (Mon): add up the variables x_i such that shift i covers Monday
- We need a way to specify elements of a set that meet certain characteristics

2 Some more set notation

- What if we only want certain elements of a set?
- ":" notation

$$j \in S : [\text{condition}] \quad \Leftrightarrow \quad j \in \text{elements of } S \text{ such that [condition] holds}$$

- For example:
  - Define $N = \{1, 2, 3\}$, $S_1 = \{a, b\}$, $S_2 = \{b, c\}$, $S_3 = \{a, c\}$
  - Then
    - $j \in N : j \geq 2 \quad \Leftrightarrow \quad \boxed{j \in \{2, 3\}}$
    - $j \in N : a \in S_j \quad \Leftrightarrow \quad \boxed{j \in \{1, 3\}}$
- Some people use "|" instead ":"
- Describe the constants of Example 1 using sets and parameters.

Sets. D = set of days = {M, T, W, Th, F, Sa, Su}
S = set of shifts = {1, 2, 3, 4, 5, 6, 7}

D_j = set of days in shift j for $j \in S$
e.g. D_1 = set of days in shift 1 = {M, T, W, Th, F}
D_4 = set of days in shift 4 = {Th, F, Sa, Su, M}

Parameters. r_i = required # employees for day i for $i \in D$

- Write a parameterized linear program for Example 1 using the sets and parameters you described above.

DVs: x_j = # employees assigned to shift j for $j \in S$

$$\min \quad \sum_{j \in S} x_j \qquad \text{(total \# employees)}$$

$$\text{s.t.} \quad \sum_{j \in S : i \in D_j} x_j \geq r_i \qquad \text{for } i \in D \qquad \text{(daily requirements)}$$

$$x_j \geq 0 \qquad \text{for } j \in S \qquad \text{(nonnegativity)}$$

$$i = M \qquad \sum_{j \in S : M \in D_j} x_j \geq r_M$$

3 The Rusty Knot, revisited

Example 2. At the Rusty Knot, tables are set and cleared by runners working 5-hour shifts that start on the hour, from 5am to 10am. Runners in these 5-hour shifts take a mandatory break during the 3rd hour of their shifts. For example, the shift that starts at 9am ends at 2pm, with a break from 11am-12pm. The Rusty Knot pays $7 per hour for the shifts that start at 5am, 6am, and 7am, and $6 per hour for the shifts that start at 8am, 9am, and 10am. Past experience indicates that the following number of runners are needed at each hour of operation:

Hour	Number of runners required
5am-6am	2
6am-7am	3
7am-8am	5
8am-9am	5
9am-10am	4
10am-11am	3
11am-12pm	6
12pm-1pm	4
1pm-2pm	3
2pm-3pm	2

Formulate a linear program that determines a cost-minimizing staffing plan. You may assume that fractional solutions are acceptable.

Sets. H = set of hours = $\{5, 6, 7, 8, 9, 10, 11, 1, 2\}$

S = set of shifts = $\{5, 6, 7, 8, 9, 10\}$

H_j = set of hours covered by shift j for $j \in S$

e.g. $H_5 = \{5, 6, 8, 9\}$

$H_9 = \{9, 10, 12, 1\}$

Parameters. r_i = # runners required in hour i for $i \in H$

c_j = hourly cost of shift j for $j \in S$

DVs. x_j = # runners assigned to shift j for $j \in S$

$$\min \quad \sum_{j \in S} (5c_j)\, x_j \qquad \text{(total cost)}$$

$$\text{s.t.} \quad \sum_{j \in S : i \in H_j} x_j \geq r_i \quad \text{for } i \in H \qquad \text{(hourly requirements)}$$

$$x_j \geq 0 \quad \text{for } j \in S \qquad \text{(nonnegativity)}$$